Downside Risk-Adjusted
Performance Measurement

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  Morningstar Associates, LLC
Agenda

- Omega, Sharpe-Omega, & the Sortino Ratio
- Kappa as a generalization
- The Lottery Test
- Skewness & kurtosis
- Johnson distributions
- Estimating Kappa
- Impact of Kappa variant choice on performance rankings
- Summary
Shadwick & Keating’s Omega

\[ \Omega(\tau) = \int_{-\infty}^{\tau} \left[ 1 - F(R) \right] dR \]

\[ \int_{\tau}^{\infty} F(R) dR \]

\( \tau \) = threshold return

\( F(.) \) = cumulative density function of returns
Example of Omega

\[ \Omega(0) = \frac{0.7532}{0.2034} = 3.70 \]
Kazemi, Schneeweis, & Gupta’s Sharpe-Omega

Target-Based Sharpe Ratio \(= \frac{\mu - \tau}{\sigma}\)

Sharpe-Omega \(= \frac{\mu - \tau}{P(1+\tau)}\)

\(\mu = \text{mean of return distribution}\)
\(\sigma = \text{standard deviation of return distribution}\)
\(P(1+\tau) = \text{price of put option (with strike price = 1+\tau)}\)
Equivalence of Omega & Sharpe-Omega

\[ \text{Sharpe-Omega} = e^{r_f} \left[ \Omega(\tau) - 1 \right] \]

\( r_f \) = continuous risk-free rate
Sortino Ratio

\[ S(\tau) = \frac{\mu - \tau}{\sqrt{\int_{-\infty}^{\tau} (\tau - R)^2 \, dF(R)}} \]
Lower Partial Moments (Downside Risk Measures)

\[ \text{LPM}_n (\tau) = \int_{-\infty}^{\tau} (\tau - R)^n \, dF(R) \]
Kappa: A Generalized Downside Risk-Adjusted Measure
(Kaplan & Knowles)

\[ K_n(\tau) = \frac{\mu - \tau}{\sqrt[n]{LPM_n(\tau)}} \]
Omega & the Sortino Ratio as Special Cases of Kappa

\( n=1 \Rightarrow \text{Omega:} \)

\[ \Omega(\tau) = K_1(\tau) + 1 \]

\( n=2 \Rightarrow \text{Sortino Ratio:} \)

\[ S(\tau) = K_2(\tau) \]
Equivalence of Kappa & Target-Based Sharpe Ratio Rankings when Returns are Normally Distributed (Di Pierro & Mosevich)
Shadwick’s Lottery Test

- Consider a lottery ticket
  - Price = $1
  - 1 out of 1,000,000 chance of paying $1,000,000

- Is it better to buy or sell this lottery ticket?
Characteristics & Ranking of Sides of Lottery Ticket

× **Characteristics**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Buy</th>
<th>Sell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>999.9995</td>
<td>999.9995</td>
</tr>
<tr>
<td>Skewness</td>
<td>999.9985</td>
<td>-999.9985</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>99999998.0000</td>
<td>99999998.0000</td>
</tr>
</tbody>
</table>

× **Rankings**

× $K_1$ (Omega) ranks Buy better than Sell for all thresholds.

× For all other values of $n$, ranking depends on threshold.

× So Sortino Ratio ($n=2$) fails lottery test.
Differences in Kappa between Sides of Lottery Ticket

![Graph showing differences in Kappa between Buy and Sell sides.]

- Kappa_1
- Kappa_2

Threshold
Skewness

The graph illustrates the effect of skewness on a probability distribution. The skewness values are 0, +0.75, and -0.75, with corresponding curves marked as S=0, S=+0.75, and S=-0.75 respectively. The distribution for S=0 is symmetrical, while S=+0.75 and S=-0.75 are skewed, with S=+0.75 having a positive skew and S=-0.75 having a negative skew.
Johnson Family of Distributions

- **Skewness**
- **Kurtosis**

- **Bounded**
- **Inadmissible**

- Normal
- 3-Param Lognormal

Unbounded vs. Bounded

- Johnson Family Distributions

Morningstar
Estimating Kappa with Johnson Distributions

- Estimate mean, standard deviation, skewness, & kurtosis of returns.

- From skewness & kurtosis, select appropriate Johnson distribution:
  - \( s \approx 0 \) & \( k \approx 3 \) \( \Rightarrow \) Normal
  - \((s, k)\) near boundary \( \Rightarrow \) 3-Parameter Lognormal
  - \((s, k)\) above boundary \( \Rightarrow \) Unbounded
  - \((s, k)\) below boundary \( \Rightarrow \) Bounded

- Calculate parameters of selected distribution.

- Calculate Kappa using numerical integration.
Example: HFR Monthly Hedge Fund Indices, 1/1990-12/2004
Returns in excess of T-bill
# Estimates of Kappa(0) for HFR Monthly Hedge Fund Indices

<table>
<thead>
<tr>
<th>Index</th>
<th>Johnson Dist.</th>
<th>Kappa&lt;sub&gt;1&lt;/sub&gt;</th>
<th>Kappa&lt;sub&gt;2&lt;/sub&gt;</th>
<th>Kappa&lt;sub&gt;3&lt;/sub&gt;</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Distressed Securities</td>
<td>Unbounded</td>
<td>2.707</td>
<td>0.909</td>
<td>0.525</td>
<td>0.501</td>
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<tr>
<td>2. Convertible Arbitrage</td>
<td>Bounded</td>
<td>2.680</td>
<td>0.947</td>
<td>0.586</td>
<td>0.547</td>
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<tr>
<td>3. Equity Market Neutral</td>
<td>Unbounded</td>
<td>2.435</td>
<td>1.029</td>
<td>0.695</td>
<td>0.479</td>
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<tr>
<td>4. Event-Driven</td>
<td>Unbounded</td>
<td>2.031</td>
<td>0.713</td>
<td>0.428</td>
<td>0.447</td>
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<tr>
<td>5. Equity Hedge</td>
<td>Unbounded</td>
<td>1.975</td>
<td>0.837</td>
<td>0.556</td>
<td>0.415</td>
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<tr>
<td>6. Merger Arbitrage</td>
<td>Bounded</td>
<td>1.799</td>
<td>0.546</td>
<td>0.313</td>
<td>0.410</td>
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<tr>
<td>7. Macro</td>
<td>Unbounded</td>
<td>1.785</td>
<td>0.832</td>
<td>0.590</td>
<td>0.394</td>
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<tr>
<td>8. Sector</td>
<td>Unbounded</td>
<td>1.323</td>
<td>0.561</td>
<td>0.362</td>
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<td>9. Fund of Funds</td>
<td>Unbounded</td>
<td>1.245</td>
<td>0.503</td>
<td>0.313</td>
<td>0.299</td>
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<tr>
<td>10. Fixed Income Arbitrage</td>
<td>Unbounded</td>
<td>1.110</td>
<td>0.401</td>
<td>0.235</td>
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<tr>
<td>11. Emerging Markets</td>
<td>Unbounded</td>
<td>0.798</td>
<td>0.334</td>
<td>0.213</td>
<td>0.223</td>
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<td>12. Short Selling</td>
<td>Unbounded</td>
<td>-0.013</td>
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Kappa-Based Ranks on HFR Monthly Hedge Fund Indices

<table>
<thead>
<tr>
<th>Index</th>
<th>Kappa_1</th>
<th>Kappa_2</th>
<th>Kappa_3</th>
<th>Sharpe</th>
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<td>1</td>
<td>3</td>
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<td>Event-Driven</td>
<td>4</td>
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<td>Merger Arbitrage</td>
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<td>Macro</td>
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<td>Fund of Funds</td>
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<tr>
<td>Short Selling</td>
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Summary

- Omega & the Sortino Ratio are special cases of Kappa.
- Sharpe-Omega is just a restatement of Omega.
- All Kappa variants give the same rankings as the Sharpe Ratio when returns are normal.
- Omega is the only version of Kappa that passes the lottery test.
- Skewness & kurtosis can be modeled using Johnson distributions.
- Johnson distributions can be used to estimate Kappa from the first four moments.
- The 3-parameter lognormal, used by Sortino & Forsey to estimate downside risk, is a special case of the Johnson distributions.
- Choice of Kappa variant affects risk-adjusted performance rankings.


